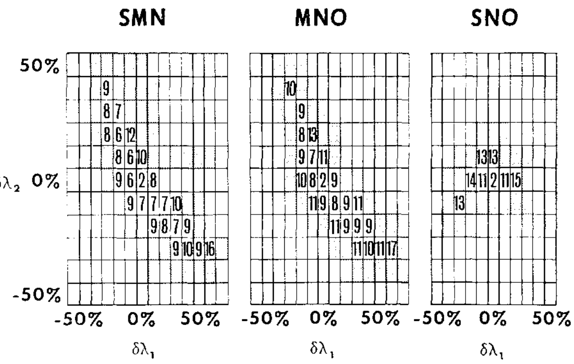

Fig. 2 Convergence envelope for MQM, final time optimal.

multipliers were normalized (MNO). It was unnecessary to consider a case with no normalization since this information could be inferred from the other three cases. In all cases, the value for the final time chosen initially was optimal as was the value for λ_{4_0}. The Lagrange multipliers were normalized by λ_{3_0}, and the optimal value for λ_{3_0} was used for the case when only the initial state data were normalized. For the unnormalized initial state data, an M-K-S system of units was used.

If the initially assumed values (Fig. 1) of λ_{1_0} and λ_{2_0} corresponding to different trial trajectories are presented such that λ_{1_0} is the abscissia and λ_{2_0} is the ordinate, a convergence envelope is formed about the optimal values of λ_{1_0} and λ_{2_0}. The coordinates for the envelopes are the deviations in percent from the optimal values of λ_{1_0} and λ_{2_0}.

Figure 1 illustrates the envelopes of convergence for MPF for the SMN, MNO, and SNO cases. It is seen that normalizing the state data reduces the number of iterations required to achieve convergence, but does not effectively change the size of the convergence envelope. By contrast, normalizing the multipliers not only reduces the number of iterations required to achieve convergence, but also increases the size of the convergence envelope. Normalizing both the state data and the multipliers produces the most favorable results. Figure 2 presents similar envelopes of convergence for MQM. The trend for MQM is the same as for MPF; however, the convergence envelope for the SNO case is slightly larger for MQM than for MPF. This is apparently the result of establishing an effective metric to be used as a convergence criteria for MQM. The initial reference solution used for MQM was the nonlinear differential equations. The physical significance of the convergence envelopes for SMN and MNO cases is explained in detail in Ref. 3.

Normalizing the state and the multipliers caused an increase in size of the convergence envelope by a factor of 700% for MPF and over 300% for MQM. In addition, the computational efficiency was improved by about 50% in terms of the number of iterations required for convergence, and a 20% reduction in computer time per iteration was achieved by not integrating the additional perturbation vector.

Conclusions

The computational advantages of embedding the initial data into the interval $[-1,1]$ have been investigated for the indirect methods of trajectory optimization. The normalization of both the state data and the Lagrange multipliers produced the most significant results, and normalizing only the multipliers gave more favorable results than normalizing only the state data.

Based on the results of this investigation, normalizing the state data and the multipliers is a significant step toward reducing the sensitivity of the initially assumed parameters for convergence characteristics of the indirect optimization methods. This improvement combined with a more sophisticated correction scheme greatly improves the possibility of convergence in one computer run. In addition, this normalization takes full advantage of the numerical procedure and could be used with equal success for direct methods.

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Simplified Technique for Estimating Navigational Accuracy of Interplanetary Spacecraft

LOUIS KINGSLAND JR.* AND WILLARD E. BOLLMAN†
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.

THE principal way to determine the position and velocity of a spacecraft on an interplanetary trajectory is by observing the Doppler shift in a coherent, two-way radio signal between Earth tracking stations and the spacecraft. By combining observations over a period of time, it is possible to estimate the trajectory of the spacecraft with considerable accuracy. It is also possible to estimate the navigational accuracy that might be obtained on potential missions by performing statistical studies with real-time orbit determination programs, but the process is very costly and time-consuming, largely because of the requirement to integrate the equations of spacecraft motion.

Attempts have been made to model the planet-spacecraft geometry by means of perfect hyperbolic motion, and while reasonably successful in reproducing the near-planet trajectory geometry, this approach has not been adequate to model the "transitional" phase of the trajectory, where the sun and the planet significantly affect the motion of the spacecraft. Much important tracking data are taken during this period, and experience has shown that a trajectory model that assumes perfectly hyperbolic spacecraft-planet motion fails to

Received August 13, 1969; presented as Paper 69-899 at the AIAA/AAS Astrodynamics Conference, Princeton, N. J., August 20–22, 1969; revision received October 6, 1969. This paper represents one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract NAS 7-100, sponsored by NASA.

* Manager for Mission Analysis and Engineering, Viking Orbiter Office.

† Member of Technical Staff, Systems Analysis Section.

produce navigational accuracy estimates comparable with those produced by real-time orbit determination programs.

This Note describes a new approach in which information from the Doppler tracking of an interplanetary spacecraft approaching a distant planet is approximated by combining the effects of three relatively simple hypothetical trajectories.

Outline of Method

First we must obtain partial derivatives, $\delta\dot{\rho}/\delta Q_i$, of the observable range rate $\dot{\rho}$, with respect to each of the parameters Q_i, to be estimated. These partial derivatives are then combined with a priori information to produce an estimate of the resulting uncertainty following a given amount of Doppler data. Except for certain significant simplifications, the proposed procedure is essentially the same as that presently used in real-time orbit determination programs to produce an estimate of navigational accuracy.[1] The elements of this procedure can be summarized as follows.

1) The partial derivatives of the range rate observable $\dot{\rho}$ are obtained, with respect to the n parameters to be estimated;

$$\phi_i = \partial\dot{\rho}/\delta Q_1, \ldots, \partial\dot{\rho}/\delta Q_n (n \times 1 \text{ vector}) \tag{1}$$

2) The vector of partials is multiplied by itself and by a weighting matrix ω_i to form the matrix

$$J_i^* = \phi_i \omega_i^{-1} \phi_i^t \tag{2}$$

3) J_i^* is added to the accumulated matrix;

$$J^* = J_1^* + J_2^* + \ldots + J_{i-1}^* \tag{3}$$

4) Given an a priori covariance matrix $\tilde{\Gamma}$, of the parameters to be estimated, the estimated covariance matrix is obtained as follows:

$$\Gamma = (J^* + \tilde{\Gamma}^{-1})^{-1} \tag{4}$$

Approximation to Partial Derivatives

When the spacecraft is far from the target planet, the behavior of the trajectory and of the partial derivatives will be essentially that of a heliocentric conic. When it is very close to encounter with the target planet, its motion will be almost perfectly hyperbolic and will be affected only slightly by heliocentric effects.

In keeping with these "boundary conditions," the following approximation to the partial derivatives is used to model their behavior throughout the encounter sequence:

$$\partial\dot{\rho}/\partial Q_j = (\partial\dot{\rho}/\partial Q_j)_{\text{helio}} + (\partial\dot{\rho}/\partial Q_j)_{\text{planet}} - (\partial\dot{\rho}/\partial Q_j)_{\Delta} \tag{5}$$

where $(\partial\dot{\rho}/\partial Q_j)_{\text{helio}}$ is calculated assuming that the planet's mass will have no effect on the spacecraft and that the spacecraft's trajectory will be heliocentric conic (see Fig. 1, heliocentric trajectory curve). $(\partial\dot{\rho}/\partial Q_j)_{\text{planet}}$ is calculated assuming that the only force affecting the motion of the spacecraft relative to the target planet is the gravitational force of the planet and that the trajectory is perfectly hyperbolic (see Fig. 1, planet-centered hyperbolic trajectory curve). $(\partial\dot{\rho}/\partial Q_j)_{\Delta}$ is calculated assuming that the spacecraft moves in a perfectly straight line, relative to the target planet, and achieves a perfect match of the boundary conditions. At encounter, this trajectory will be identical to the heliocentric trajectory and, far from the planet, will be identical to that of the planet-centered hyperbolic trajectory (Fig. 1).

Hamilton and Melbourne[2] examined the information content of a single pass of Doppler data and derived an approximate equation that accurately describes the observed instantaneous range rate of a distant spacecraft and that shows the effect of the earth's rotation on Doppler data;

$$\dot{\rho} = \dot{r} + \omega r_s \cos\delta \sin\omega(t - t_z) \tag{6}$$

where $\dot{\rho}$ is the range rate from the tracking station to the spacecraft; $\dot{r}$ is the range rate from the center of the Earth to the spacecraft; r_s is the distance of the tracking station off the earth's spin axis; ω is the rotation rate of the Earth; δ is the geocentric equatorial declination of the spacecraft; t is the time of the observation; and t_z is the time of maximum elevation (zenith).

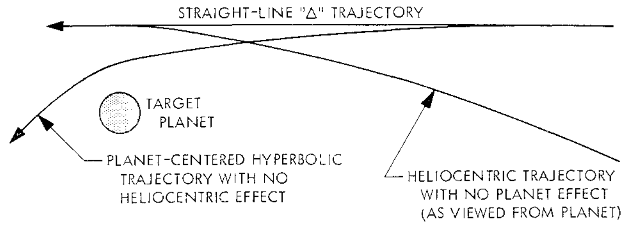

Fig. 1. Hypothetical trajectories used in approximation partial derivatives of range rate.

In order to further simplify the execution of several of the steps described previously, the approximate data equation (Eq. 6) is assumed to be a sufficiently accurate representation of the observable to be used in the calculation of partial derivatives. This results in partial derivatives of the form

$$\partial\dot{\rho}/\partial Q_j = a_j + b_j \cos\xi + c_j \sin\xi \tag{7}$$

where $\xi \equiv \omega(t - t_z)$; $a_j = \partial\dot{r}/\partial Q_j$; $b_j = \omega r_s \cos(\partial\xi/\partial Q_j)$; and $c_j = -\omega r_s \sin\delta(\partial\delta/\partial Q_j)$.

Calculation of Partial Derivatives

The partial derivatives of geocentric range rate $\dot{r}$ are computed as follows:

$$\dot{r} = \dot{\mathbf{r}}\cdot\mathbf{r}/r = (\dot{\mathbf{R}}_s - \dot{\mathbf{R}}_E)(\mathbf{R}_s - \mathbf{R}_E)/|\mathbf{R}_s - \mathbf{R}_E| \tag{8}$$

$$\partial\dot{r}/\partial x_i = (\dot{x}_i/r) - (\dot{r}x_i/r^2),\ \partial\dot{r}/\partial\dot{x}_i = x_i/r \tag{9}$$

where $\mathbf{R}_s$ is the sun-spacecraft vector; $\mathbf{R}_E$ is the sun-Earth vector; x_1, x_2, x_3 are the ecliptic components of $\mathbf{r}$; and $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ are the ecliptic components of $\dot{\mathbf{r}}$.

The partial derivatives of geocentric equatorial declination are computed as follows:

$$\sin\delta = (x_2 \sin\gamma + x_3 \cos\gamma)/r \tag{10}$$

$$\cos\delta = (1 - \sin^2\delta)^{1/2} \tag{11}$$

$$\partial\delta/\partial x_i = (\cos\delta)^{-1}(\partial \sin\delta/\partial x_i),\ \partial\delta/\partial\dot{x}_i = 0 \tag{12}$$

where γ is the angle of inclination between the ecliptic and equatorial planes.

The partial derivative of ξ is computed by first observing that

$$\xi \equiv \omega(t - t_z) = \omega t_\mu + \alpha_s + \lambda_s - \alpha \tag{13}$$

where t_μ is the true universal time; α_s is the instantaneous right ascension of the mean sun; λ_s is the longitude of the tracking station measured eastward from Greenwich; and α is the geocentric equatorial right ascension of the spacecraft.

This leads to the following simplifications in the partial derivatives:

$$\left.\begin{aligned} \partial\xi/\partial x_i &= -\partial\alpha/\partial x_i = (-1/1 + \tan^2\alpha)(\partial \tan\alpha/\partial x_i) \\ &[\text{where } \tan\alpha = (x_2 \cos\gamma - x_3 \sin\gamma)/x_1] \\ \partial\xi/\partial\dot{x}_i &= -\partial\alpha/\partial\dot{x}_i = 0 \end{aligned}\right\} \tag{14}$$

$$\partial\dot{\rho}/\partial r_s = \omega \cos\delta \sin\xi \tag{15}$$

It is slightly more convenient to express station longitude uncertainty in meters of east-west distance d_L rather than angles;

$$\partial\dot{\rho}/\partial d_L = \partial\dot{\rho}/r_s\partial\lambda_s = \omega \cos\delta \cos\xi \tag{16}$$

Calculation of Estimate

Given the partial derivatives corresponding to a specific observation time, the $J_i^{*\prime}$ matrix is calculated, using a

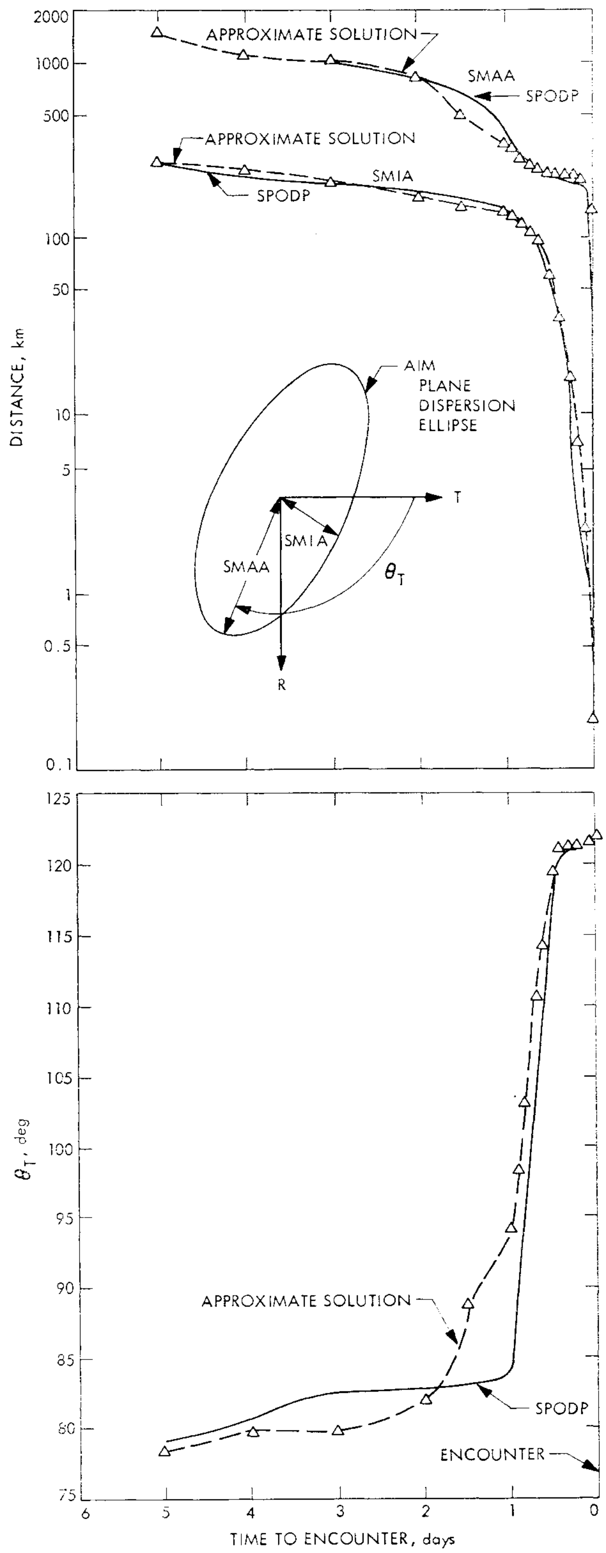

Fig. 2 Size and orientation of semimajor axes (SMAA) and semiminor axes (SMIA): Venus '67.

weighting matrix ω_i;

$$J_i^{*\prime} = \phi_i \omega_i^{-1} \phi_i^T \tag{17}$$

where ϕ_i is an $n \times 1$ vector of the partial derivatives of the observable $\dot{\rho}$, with respect to the n parameters to be estimated;

$$\omega_i = \sigma \dot{\rho}^2 \tag{18}$$

where $\sigma\dot{\rho}$ is the Doppler data noise corresponding to a 60-sec

Table 1 Encounter parameters and a priori uncertainties for representative missions; two tracking stations

Mission	Venus '67	Mars '69	Mars '73
b, km	25,000	7,375	8,023
V_∞, km/sec	3.113	6.44	2.85
θ, deg	−32	0	357.2
$\sigma_{x,y,z}$, km	1,000	10,000	500
$\sigma_{\dot{x},\dot{y},\dot{z}}$, km/sec	1	10	0.1
σ_{r_s}, m	24	10	1.5
σ_{d_L}, m	48	9.6	2.88
Data noise (per minute), mm/sec	5	3	1
Tracking from (days before encounter)	9	5	30

data arc. Each term of the $J_i^{*\prime}$ matrix will have the form

$$(J_i^{*\prime})_{jk} = [a_j a_k + (a_j b_k + a_k b_j) \cos\xi + b_j b_k \cos^2\xi + (a_j c_k + a_k c_j) \sin\xi + (b_j c_k + b_k c_j) \cos\xi \sin\xi + c_j c_k \sin^2\xi]/\sigma\dot{\rho}^2 \tag{19}$$

The effect of adding each term of the $J_i^{*\prime}$ matrix to the accumulating J^* matrix can be approximated by assuming that the coefficients a_j, b_j, and c_j will be very nearly constant throughout the tracking pass. The data can then be "compressed" by integrating over the duration of the tracking pass;

$$(J_i^{*\prime})_{jk} \simeq \int_{t_I}^{t_F} [J_i^*(t)]_{jk}' \, dt \tag{20}$$

Integration of the sine and cosine terms produces constants which depend on the tracking pass duration. If the pass is symmetrical ($t_F - t_z = t_z - t_I \equiv \Psi/\omega$), the expression for "compressed" $(J_i^{*\prime})_{jk}$ can be integrated to produce the following expression:

$$(J_i^{*\prime})_{jk} = (t_F - t_I)/\sigma\dot{\rho}^2 \{a_j a_k + (a_j b_k + a_k b_j)(\sin\Psi/\Psi) + b_j b_k/2[1 + (\sin\Psi \cos\Psi/\Psi)] + c_j c_k/2[1 - (\sin\Psi \cos\Psi/\Psi)]\} \tag{21}$$

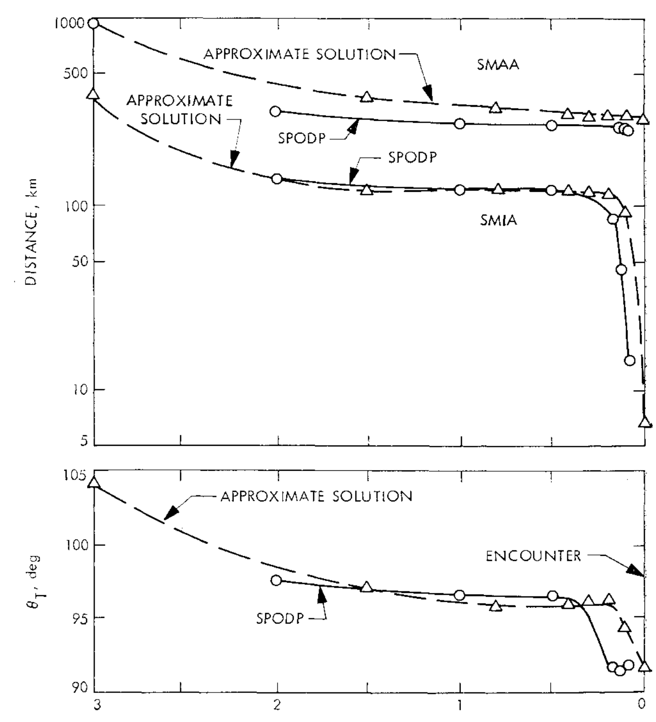

Fig. 3 Size and orientation of SMAA and SMIA: Mars '69.

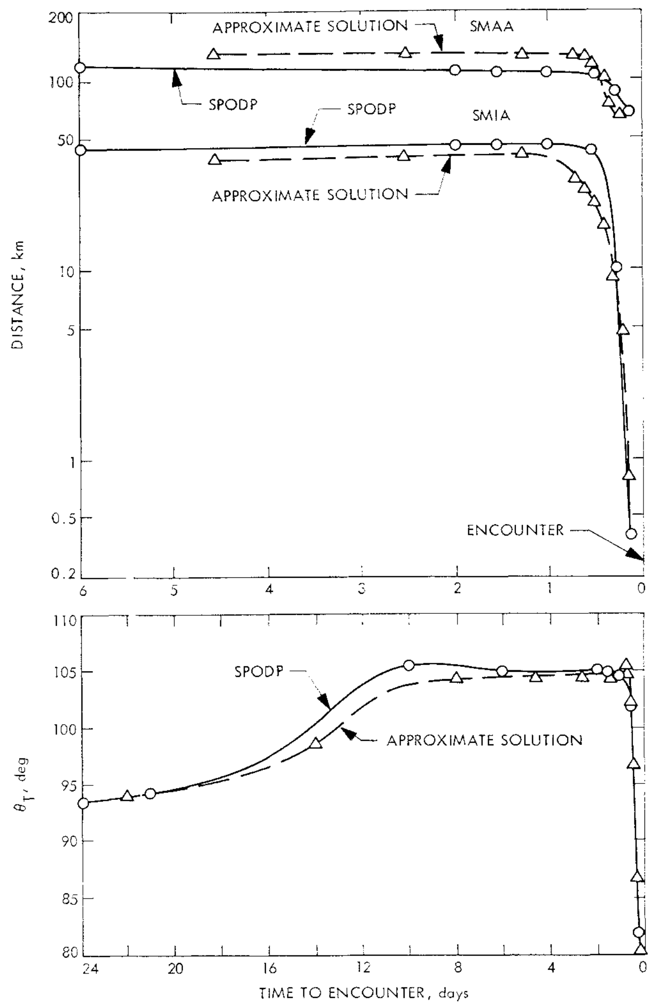

Fig. 4 Size and orientation of SMAA and SMIA: Mars '73.

where $t_F - t_I$ is the duration of the t acking pass (normally 8 to 12 hr). The coefficients a_j, b_j, and c_j vary with time as the spacecraft approaches the planet; they are, therefore, computed at sufficiently frequent intervals throughout the approach phase so that the J^* matrix is reasonably well approximated. The foregoing expression for $(J_i^{*\prime})_{jk}$ represents the "average" data from one station during the time it is tracking the spacecraft, but must be adjusted for the fact that there may be more than one station used and that no one station can track the spacecraft 24 hr/day. Therefore, the final form of J_i^* is adjusted to reflect these considerations as follows:

$$J_i^* = (N/24)J_i^{*\prime} \tag{22}$$

where N is the number of tracking stations (1, 2, or 3).

Results

The results of the approximate technique will be compared with the results of an integrated, real-time, single precision orbit determination program (SPODP) for three representative trajectories: Venus '67, Mars '69, and Mars '73. Identical assumptions and a priori uncertainties were used for both approximate and exact calculations, and are summarized in Table 1.

The results for Venus '67 (Fig. 2) indicate good agreement between the SPODP and the approximate solution. The major difference is that the SPODP indicates a fairly sharp transition in the uncertainty ellipse at encounter minus 1 day, while the approximate solution results in a somewhat earlier and more gradual transition from initial to final orientation of the uncertainty ellipse. In general, the size, shape, and orientation of the uncertainty ellipse in the aim plane are remarkably well approximated and are certainly within the accuracy required for most guidance and navigation studies. For Mars '69 (Fig. 3), the size and orientation of the dispersion ellipse are fairly well modeled, except that the transition from initial to final orientation is slightly delayed.

The selected trajectory for a 1973 Mars mission would permit an arrival at Mars on February 14, 1974. This trajectory is presented because it presently represents the "worst" case encountered among the various comparisons made between the SPODP and the approximate solution. Figure 4 shows that the estimates of the semimajor axes provide a fair approximation to SPODP results. The orientation angle of the dispersion ellipse, however, is reproduced with very remarkable accuracy.

In conclusion, the combination of three hypothetical trajectories in the calculation of approximate partial derivatives appears to have been successful in modeling aspects of a three-body trajectory which previously required numerical inregration. This approximate solution technique described in this Note should be of great value in performing navigational accuracy studies for interplanetary spacecraft trajectories. Further extension of this modeling procedure into other three-body trajectory problems also may be possible.

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ICRPG Liquid Propellant Thrust Chamber Performance Evaluation Methodology

WALTER B. POWELL*
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.

Development of the Methodology

THE Performance Standardization Working Group (PSWG) of the Inter-Agency Chemical Rocket Propulsion Group (ICRPG) was organized in 1965 to develop a methodology for the experimental determination, analytical prediction, correlation, extrapolation, and flight confirmation of the performance of liquid propellant rocket engines. The Working Group Steering Committee established three working committees: Over-all Concepts, Theoretical Methods, and Experimental Methods. To define a useful objective which could be accomplished within a finite time, the initial effort was

Presented as Paper 69-468 at the AIAA 5th Propulsion Joint Specialist Conference, U.S. Air Force Academy, Colo., June 9–13, 1969; submitted August 25, 1969; revision received October 20, 1969. This Note summarizes the work done from 1965 through 1968 by the Overall Concepts and Theoretical Methods Committees cited in the text. The members of these committees, too numerous to identify by name, were drawn from industry and the government agencies, and deserve to share equally the credit for these accomplishments, as they shared in the labors of this cooperative effort.

* Member of the Technical Staff, Liquid Propellant Section. Associate Fellow AIAA.